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RECENT PROGRESS ON MATRIX COMPLETION PROBLEMS

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Abstract

Matrix Completion Problems are an important branch of research on Matrix Theory. In the last decades, many authors have studied this type of questions. In this paper we mention the most important results available in the literature, concerning the prescription of the characteristic polynomial of a square matrix partitioned into 2×2 blocks, when some of its blocks are fixed and the others vary. We still present our contribution in this area. In general, the solution of this type of problems contemplates necessary and sufficient conditions with an expected form (involving the interlacing inequalities for the invariant factors). However, in a particular case, we show that the "expected" condition is not necessary. For that purpose let us consider n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$ and let $C = [C_{i,j}] \in F^{n \times n}$ be a partitioned matrix, where the blocks $C_{i,j} \in F^{p_i \times p_j}$, $i, j \in \{1, 2, 3\}$. Let f be a monic polynomial of degree n and suppose that the blocks $C_{1,1}, C_{1,2}, C_{3,1}$ are prescribed and the others are free. Let $\alpha_1 | \cdots | \alpha_{p_1}$ be the invariant factors of the matrix pencil $[xI_{p_1} - C_{1,1} - C_{1,2}]$ and let $\beta_1 | \cdots | \beta_{p_1}$ be the invariant factors of

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$$\begin{bmatrix} xI_{p_1} & -C_{1,1} \\ & -C_{3,1} \end{bmatrix}.$$

In this paper we show that the following divisibility condition

$$\alpha_1 \cdots \alpha_{p_1 - p_3} \beta_1 \ldots \beta_{p_1 - p_2} | f$$

is not a necessary condition for the existence of a matrix C with prescribed form and prescribed characteristic polynomial f.

1. Introduction

In the last decades, many authors have studied a particular case of the Matrix Inverse Problems, the so-called *Matrix Completion Problems*.

The inverse problems arise in a variety of situations in our lives and their nature consists in deducing a cause from an effect. In general, an inverse problem can be described as a problem where the answer is known, but not the question. For example, it can be calculated easily how long it will take to boil a given quantity of water. Now suppose we know how long a liquid takes to boil, can we identify the liquid? This is an inverse problem. It should be pointed out that this problem becomes more complex, since it is possible that different liquids share the same specific heat capacity.

A special class of inverse problems, are the Inverse Eigenvalue Problems. It seems that the research on Inverse Eigenvalue Problems began in 1933, with the work of Krein [14] with applications to mechanics.

A general goal in such type of questions, is to establish necessary and sufficient conditions under which it is possible to describe the eigenvalues of the required matrix. The proof of these conditions can be very hard.

As an example we will refer to an old and "classic" problem, proposed in 1949 by Suleimanova [28]. The author had the purpose of describing the possible eigenvalues of a positive stochastic matrix. The partial answer obtained by the author is presented in the following result.

Theorem 1 [28]**.** *Let F be the field of real numbers. Let $c_1, \ldots, c_n \in F$ such that $c_1 = 1, |c_j| < 1,$ for all $j \in \{2, \ldots, n\}$. If the sum of all*

$|c_j|$, $j \in \{2, ..., n\}$ *over those* $c_j < 0$ *is less than* 1, *then there exists an* $n \times n$ *positive stochastic matrix with eigenvalues* $c_1, ..., c_n$. *If all* $c_j < 0$, $j \in \{2, ..., n\}$ *the condition is also necessary*.

We believe that this Theorem is one of the earliest results in Inverse Eigenvalue Problems. The complete answer to the previous problem is not yet found.

More generally, the Matrix Inverse Problems (which includes the Inverse Eigenvalue Problems) consist in studying the existence of a matrix (or a combination of matrices) satisfying certain properties. They arose from a remarkable variety of applications into many areas, including systems and control theory, control design, geophysics, particle physics, circuit theory, and so on. In particular, the Matrix Completion Problems, a subclass of the Matrix Inverse Problems, consist in studying the possibility to "complete" a matrix, when some of its entries are prescribed (i.e., are fixed), such that the resulting matrix satisfies certain properties. In this context "to complete" means to attribute values to the remaining entries. Basically, the structure of the Matrix Completion Problems is the following: given a matrix and a part of the given matrix (as a submatrix, some entries, so on) the aim of these problems is to describe conditions for which we can fill the unknown entries, such that the resulting matrix satisfies the required properties.

2. Background

In this section we will introduce some notation and background needed for the rest of the paper.

Let F be a field.

The ring of the polynomials in the indeterminate x, with coefficients in F, is denoted by $F[x]$.

If $f(x) \in F[x]$, the degree of $f(x)$ is denoted by $\deg(f(x))$. Make convention that $\deg(0) = -\infty$.

Let $D = F$ or $D = F[x]$ and let m, n be positive integers. We denote by $D^{m \times n}$ the set of all matrices in D of type $m \times n$, i.e., with m rows and

n columns.

If $A \in D^{m \times m}$ we denote by tr A its trace.

The symbol $|$ is used in the following way: if $f(x),\ g(x) \in F[x]$, then $f(x) | g(x)$ means "$f(x)$ divides $g(x)$".

Now we present some definitions and results that can be found in many books on Linear Algebra, see for example [10, 12, 13, 15, 16, 18].

Definition 2. Let $p(x),\ q(x) \in F[x]$. The polynomials $p(x)$ and $q(x)$ are associates in $F[x]$ if $p(x) | q(x)$ and $q(x) | p(x)$.

Remark 1. In the previous conditions the polynomials $p(x)$ and $q(x)$ are associates in $F[x]$ if and only if there exists $u \in F \setminus \{0\}$ such that $q(x) = up(x)$.

The relation "to be associate" is an equivalence relation on $F[x]$.

Let R be the set of all monic polynomials and the zero polynomial. This set R is a representative set for the equivalence classes for this relation and it is closed for the product.

Definition 3. Let $A(x) \in F[x]^{m \times n}$. The greatest common divisor chosen in R, of the determinants of the submatrices of size $k \times k$ of $A(x)$, $k \in \{1, \ldots, \min\{m, n\}\}$ is denoted by $d_k(x)$. If $k \leq \operatorname{rank} A(x)$, we say that $d_k(x)$ is the k-th determinantal divisor of $A(x)$. Make convention that $d_0(x) = 1$.

It is known that if $A(x) \in F[x]^{m \times n}$ and rank $A(x) = r$, then

(i) $d_k(x) \neq 0$ if and only if $k \leq r$;

(ii) $d_{k-1}(x) | d_k(x),\ k \in \{1, \ldots, r\}$.

Definition 4. The k-th invariant factor of $A(x)$ is the element

$$i_k(x) = \frac{d_k(x)}{d_{k-1}(x)},\ k \in \{1, \ldots, \operatorname{rank}\ A(x)\},$$

with the convention that $i_0(x) = 1$.

Note that according to the previous definitions, the determinantal divisors and the invariant factors of the matrix $A(x)$ are monic polynomials.

It is known that $i_{k-1}(x) | i_k(x),\ k \in \{1, \ldots, \text{rank } A(x)\}$.

It is also known that, if $A(x), B(x) \in F[x]^{m\times n}$ are equivalent matrices, then $A(x)$ and $B(x)$ have the same determinantal divisors and the same invariant factors.

The following theorem describes a canonical form for the polynomial matrices equivalence and it is known as The Smith Normal Form.

Theorem 5. *Let* $A(x) \in F[x]^{m\times n}$. *Then* $A(x)$ *is equivalent to a unique matrix of the form*

$$\left[\begin{array}{c|c} diag(i_1(x), \ldots, i_r(r)) & 0 \\ \hline 0 & 0 \end{array}\right],$$

where $r = \text{rank } A(x), i_1(x), \ldots i_r(x)$ *are monic polynomials and* $i_j(x) | i_{j+1}(x),\ j \in \{1, \ldots, r-1\}$. *The elements* $i_1(x), \ldots, i_r(x)$ *are the invariant factors of* $A(x)$.

As a consequence of this theorem it follows that:

(i) $A(x), B(x) \in F[x]^{m\times n}$ are equivalent if and only if they have the same invariant factors.

(ii) $A(x), B(x) \in F[x]^{m\times n}$ are equivalent if and only if they have the same determinantal divisors.

Let F be a field and let $A \in F^{m\times m}$.

The polynomial matrix $xI_m - A$ is called the *characteristic matrix* of A and its determinant is called the *characteristic polynomial* of A.

The invariant factors of $xI_m - A$ are called the *invariant polynomials* of A.

Note that the matrix $xI_m - A$ has rank m, since its determinant is different from zero. Consequently A has m invariant polynomials,

$$f_1(x)|\cdots|f_m(x).$$

It is also known that the characteristic polynomial of a matrix $A \in F^{m\times m}$ is equal to the product of its invariant polynomials.

Remark 2. $A, B \in F^{m\times m}$ are similar matrices in F if and only if they have the same invariant polynomials.

Let $A \in F^{m\times m}$. The following result is known as the *Natural Form* of A.

Theorem 6. *Let* $A \in F^{m\times m}$ *and* $f_1(x), \ldots, f_r(x) \in F[x]$ *be monic polynomials, different from* 1, *such that* $f_1(x)|\cdots|f_r(x)$. *Then A is similar to*

$$C(f_1) \oplus \cdots \oplus C(f_r)$$

if and only if $f_1(x), \ldots, f_r(x)$ *are the nonconstant invariant polynomials of* A.

3. Matrix Completion Problems

An important question in Matrix Completion Problems, was proposed by Oliveira [20].

Problem [20]**.** Let F be a field. Let n, p, q be positive integers such that $n = p + q$. Let $f(x) \in F[x]$ be a monic polynomial of degree n. Let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \tag{1}$$

be a partitioned matrix, where $A_{1,1} \in F^{p\times p}$, $A_{2,2} \in F^{q\times q}$. Suppose that some of the blocks $A_{i,j}$, $i, j \in \{1, 2\}$, are known. Under which conditions does there exist a matrix of the form (1) with characteristic polynomial $f(x)$?

Note that this problem gives rise to essentially seven distinct problems, according to the prescription of some blocks of A:

(P_1) $A_{1,1}$ prescribed;

(P_2) $A_{1,2}$ prescribed;

(P_3) $A_{1,1}$ and $A_{1,2}$ prescribed;

(P_4) $A_{1,1}$ and $A_{2,2}$ prescribed;

(P_5) $A_{1,2}$ and $A_{2,1}$ prescribed;

(P_6) $A_{1,1}$, $A_{1,2}$ and $A_{2,2}$ prescribed;

(P_7) $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$ prescribed.

The complete answer to problem (P_1), was established by Oliveira [19].

Theorem 7 [19]**.** *Let F be an arbitrary field. Let $A_{1,1} \in F^{p\times p}$ and let $f(x) \in F[x]$ be a monic polynomial of degree n. Let $f_1(x)|\cdots|f_p(x)$ be the invariant polynomials of $A_{1,1}$. Then there exit $A_{1,2} \in F^{p\times q}$, $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$ such that the matrix of the form* (1) *has characteristic polynomial $f(x)$ if and only if $f_1(x)\cdots f_{p-q}(x)|f(x)$. (Make convention that $f_1(x)\cdots f_{p-q}(x) = 1$ if $p - q \le 1$).*

Concerning problem (P_2), Oliveira also presented the complete answer in [20].

Theorem 8 [20]**.** *Let F be an arbitrary field. Let $A_{1,2} \in F^{p\times q}$ and let $f(x) \in F[x]$ be a monic polynomial of degree n. If $A_{1,2} \neq 0$, then there exist $A_{1,1} \in F^{p\times p}$, $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$ such that the matrix of the form* (1) *has characteristic polynomial $f(x)$. If $A_{1,2} = 0$, then there exist $A_{1,1} \in F^{p\times p}$, $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$ such that the matrix of the form*

(1) *has characteristic polynomial* $f(x)$ *if and only if* $f(x)$ *has a divisor of degree p.*

In [30] Wimmer gave the complete answer to problem (P_3).

Theorem 9 [30]**.** *Let F be an arbitrary field. Let* $A_{1,1} \in F^{p\times p}$, $A_{1,2} \in F^{p\times q}$ *and let* $f(x) \in F[x]$ *be a monic polynomial of degree n. Let* $f_1(x) | \cdots | f_p(x)$ *be the invariant factors of*

$$[xI_p - A_{1,1} \quad - A_{1,2}]. \tag{2}$$

Then there exist $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$ *such that the matrix of the form* (1) *has characteristic polynomial* $f(x)$ *if and only if* $f_1(x) \cdots f_p(x) | f(x)$.

Problem (P_4) has only some partial answers due to work developed by Oliveira in [20, 22, 23] and Silva in [26]. The approach used by Silva gives a complete answer for algebraically closed fields and provides a complete description of the eigenvalues of (1), when $A_{1,1}$, $A_{2,2}$ are prescribed. Note that the prescription of the characteristic polynomial is more general because it includes the situation where the eigenvalues of the matrix are outside of the field *F*. Clearly, if all the roots of $f(x)$ are in *F*, the description of the characteristic polynomial of a matrix of the form (1) coincides with the study of the possible eigenvalues of (1). In particular, if *F* is an algebraically closed field, the previous problem simply consists in studying the list of the eigenvalues of a matrix of the form (1) for the correspondent prescription of blocks.

Theorem 10 [26]**.** *Let F be an arbitrary field and let* $c_1, \ldots, c_n \in F$. *Let* $A_{1,1} \in F^{p\times p}$ *and* $A_{2,2} \in F^{q\times q}$. *Let* $f_1(x) | \cdots | f_p(x)$ *be the invariant polynomials of* $A_{1,1}$. *Suppose that* $p \geq q$. *Then there exist* $A_{1,2} \in F^{p\times q}$ *and* $A_{2,1} \in A^{q\times p}$ *such that the matrix of the form* (1) *has eigenvalues* $c_1, \ldots, c_n$ *if and only if the following conditions are satisfied*:

(a) $\operatorname{tr} A_{1,1} + \operatorname{tr} A_{2,2} = c_1 + \cdots + c_n$.

(b) *If* $p > q$, *then* $f_1(x) \cdots f_{p-q}(x) \mid (x - c_1) \cdots (x - c_n)$.

(c) *One of the following conditions is satisfied*:

(c_1) *At least one of the matrices* $A_{1,1}$, $A_{2,2}$ *is nonscalar*.

(c_2) $A_{1,1} = aI_p$ *and* $A_{2,2} = bI_q$, *with* $a, b \in F$, *and there is a permutation* σ *of* $\{1, 2, \ldots, n\}$ *such that*:

(i) $c_{\sigma(2i-1)} + c_{\sigma(2i)} = a + b$ *for* $1 \leq i \leq q$;

(ii) $c_{\sigma(j)} = a$ *for* $2q < j \leq n$.

As in the previous situation, when F is an arbitrary field, problem (P_5) only has partial solutions, established by Oliveira in [21], Silva in [25] and also by Marques and Silva in [17]. If F is an algebraically closed field, then from the following result obtained by Friedland in [9], it follows that there always exists a matrix of the form (1) with prescribed characteristic polynomial. In [9] the author described the list of the eigenvalues of a matrix of the form

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}, \tag{3}$$

over an algebraically closed field, when all $n^2 - n$ nonprincipal entries are prescribed.

Theorem 11 [9]. *Let* F *be an algebraically closed field. Let* $c_1, \ldots, c_n \in F$ *and* $(i_1, j_1), \ldots, (i_{n^2-n}, j_{n^2-n})$ *be distinct elements of* $\{1, \ldots, n\} \times \{1, \ldots, n\}$, *such that* $i_l \neq j_l$, $l \in \{1, \ldots, n^2 - n\}$. *Let* $b_{i_l, j_l} \in F$, $l \in \{1, \ldots, n^2 - n\}$. *Then there exists a matrix of the form* (3) *with eigenvalues* $c_1, \ldots, c_n$ *such that* $a_{i_l, j_l} = b_{i_l, j_l}$, $l \in \{1, \ldots, n^2 - n\}$.

Oliveira in [21] established a result when the partitioned matrix (1) has all its blocks $(A_{i,j}, i, j \in \{1, 2\})$ with the the same size, i.e., $p = q$.

Theorem 12 [21]. *Let F be an arbitrary field. Let* $A_{1,2} \in F^{p \times q}$ *and* $A_{2,1} \in F^{q \times p}$. *Suppose that* $p = q$. *Let* $c_1, \ldots, c_p \in F$. *Let* $f(x) \in F[x]$ *be a monic polynomial of degree n such that*

$$f(x) = g(x)(x - c_1) \cdots (x - c_p),$$

where $g(x) \in F[x]$. *Then there exist* $A_{1,1} \in F^{p \times p}$, $A_{2,2} \in F^{q \times q}$ *such that the matrix of the form* (1) *has characteristic polynomial* $f(x)$.

Later, Silva in [25] improved the previous result, removing the condition $p = q$. Nevertheless the author still considered a special factorization of the polynomial $f(x)$.

Remark 3 [25]. If $p = q = 1$, the answer of the problem is simple. Assuming that $A_{1,2} = [b]$ and $A_{2,1} = [c]$, and $f(x) = x^2 + dx + e$, there exist $u, v \in F$ such that

$$\begin{bmatrix} u & b \\ c & v \end{bmatrix}$$

has characteristic polynomial $f(x)$ if and only if the equation $x^2 + dx + bc + e = 0$ has a root in F.

It is also obvious that if $A_{1,2} = 0$ or $A_{2,1} = 0$, then there exist $A_{1,1} \in F^{p \times p}$, $A_{2,2} \in F^{q \times q}$ such that the matrix of the form (1) has characteristic polynomial $f(x)$ if and only if $f(x)$ has a divisor of degree p.

Theorem 13 [25]. *Let F be an arbitrary field. Let* $A_{1,2} \in F^{p \times q}$ *and* $A_{2,1} \in F^{q \times p}$. *Suppose that* $p \geq q$, $p + q \geq 2$. *Let* $c_1, \ldots, c_q \in F$. *Let* $f(x) \in F[x]$ *be a monic polynomial such that*

$$f(x) = g(x)h(x)(x - c_1) \cdots (x - c_q),$$

where $g(x), h(x) \in F[x]$, $\deg(g(x)) = q$ *and* $\deg(h(x)) = p - q$. *Then there* *exist* $A_{1,1} \in F^{p \times p}$, $A_{2,2} \in F^{q \times q}$ *such that the matrix of the form* (1) *has characteristic polynomial* $f(x)$.

From this result, Silva still obtained the complete answer for the prescription of the eigenvalues of (1), for the same prescription of blocks.

Corollary 14 [25]**.** *Let F be an arbitrary field and let* $c_1, \ldots, , c_n \in F$. *Let* $A_{1,2} \in F^{p \times q}$ *and* $A_{2,1} \in F^{q \times p}$. *Then there exist matrices* $A_{1,1} \in F^{p \times p}$, $A_{2,2} \in F^{q \times q}$ *such that the matrix of the form* (1) *has eigenvalues* $c_1, \ldots, c_n$ *if and only if one of the following conditions is satisfied*:

(i) $p \neq 1$ *or* $q \neq 1$.

(ii) $p = q = 1$, *and the equation*

$$x^2 + (c_1 + c_2)x + bc + c_1 c_2 = 0$$

has a root in F, where $A_{1,2} = [b]$ *and* $A_{2,1} = [c]$.

In [17] Marques and Silva established new sufficient conditions for the existence of a matrix of the form (1) with characteristic polynomial $f(x)$ and prescribed blocks $A_{1,2}$ and $A_{2,1}$.

In the next Theorem, the authors identified the solution for the case where the prescribed blocks are a row and a column matrix, i.e., $p = 1$ or $q = 1$. Without loss of generality, assume that $q = 1$.

Theorem 15 [17]**.** *Let F be an arbitrary field and let* $f(x) \in F[x]$ *be a monic polynomial of degree n. Let* $A_{1,2} \in F^{p \times q}$ *and* $A_{2,1} \in F^{q \times p}$. *If* $q = 1$, *then there exist matrices* $A_{1,1} \in F^{p \times p}$ *and* $A_{2,2} = [a]$, $a \in F$, *such that the matrix of the form* (1) *has characteristic polynomial* $f(x)$.

In the following result, the authors studied the general case where p, q are arbitrary and $f(x)$ may not have roots in F. Nevertheless, they still considered a factorization of the polynomial $f(x)$.

Theorem 16 [17]**.** *Let F be an arbitrary field. Let $A_{1,2} \in F^{p \times q}$ and $A_{2,1} \in F^{q \times p}$. Let $f(x) \in F[x]$ be a monic polynomial of degree n such that $f(x) = f_1(x) f_2(x)$, where $f_1(x)$ has degree p. Let*

$$t = \max\{\text{rank } A_{1,2}, \text{ rank } A_{2,1}\}.$$

If one of the following conditions is satisfied, then there exist $A_{1,1} \in F^{p \times p}$, $A_{2,2} \in F^{q \times q}$ such that the matrix of the form (1) *has characteristic polynomial $f(x)$:*

(a) $t > 1$,

(b) $t = 1$ *and* $p \neq q$,

(c) $t = 1$, *and one of the polynomials $f_1(x)$ or $f_2(x)$ is reducible.*

Note that Corollary 14, established by Silva in [25], can be obtained as a consequence of this theorem.

Notice that if none of the previous conditions $(a) - (c)$ holds, it is not always true that there exist matrices $A_{1,1} \in F^{p \times p}$ and $A_{2,2} \in F^{q \times q}$ such that (1) has characteristic polynomial $f(x)$. For example [17], if $F = \{0, 1\}$, $p = q = 2$, $f(x) = x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 + x + 1)$, and

$$A_{1,2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; A_{2,1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

it can be seen with direct calculus that there exist no matrices $A_{1,1} \in F^{p \times p}$ and $A_{2,2} \in F^{q \times q}$, such that (1) has characteristic polynomial $f(x)$.

Problem (P_6) has only a partial solution due to work developed by Silva in [27].

Theorem 17 [27]**.** *Let F be an arbitrary field. Let $A_{1,1} \in F^{p \times p}$, $A_{1,2} \in F^{p \times q}$, $A_{2,2} \in F^{q \times q}$ and let $c_1, \ldots, c_n \in F$. Let $f_1(x) | \cdots | f_p(x)$ be* *the invariant factors of* (2) *and let* $g_1(x) | \cdots | g_q(x)$ *be the invariant factors of*

$$\begin{bmatrix} -A_{1,2} \\ xI_q - A_{2,2} \end{bmatrix}.$$

Then there exists $A_{2,1} \in F^{q \times p}$ *such that the matrix of the form* (1) *has eigenvalues* $c_1, \ldots, c_n$ *if and only if the following conditions are satisfied*:

(a) $\operatorname{tr} A_{1,1} + \operatorname{tr} A_{2,2} = c_1 + \cdots + c_n$.

(b) $f_1(x) \cdots f_p(x) g_1(x) \cdots g_q(x) | (x - c_1) \cdots (x - c_n)$.

(c) *One of the following conditions is satisfied*:

(c_1) *For all* $\nu \in F$, $A_{1,1}A_{1,2} + A_{1,2}A_{2,2} \neq \nu A_{1,2}$.

(c_2) *If there exists* $\nu \in F$, *such that* $A_{1,1}A_{1,2} + A_{1,2}A_{2,2} = \nu A_{1,2}$, *then there exists a permutation* σ *of* $\{1, \ldots, n\}$ *such that*

$$c_{\sigma(2i-1)} + c_{\sigma(2i)} = \nu,$$

for all $i \in \{1, \ldots, r\}$, *where* $r = \operatorname{rank} A_{1,2}$ *and*

$$c_{\sigma(2r+1)}, \ldots, c_{\sigma(n)}$$

are roots of $f_1(x) \cdots f_p(x) g_1(x) \cdots g_q(x)$.

It is clear that this result is a complete answer for algebraically closed fields and describes completely the possible eigenvalues of (1), for the prescription of $A_{1,1}$, $A_{1,2}$ and $A_{2,2}$.

Concerning problem (P_7) we do not know any reference with nontrivial results. Clearly, if $A_{1,2} = 0$ or $A_{2,1} = 0$, then there exists $A_{2,2} \in F^{q \times q}$ such that the matrix of the form (1) has characteristic polynomial $f(x)$ if and only if the characteristic polynomial of $A_{1,1}$ divides $f(x)$.

A natural question that arises is the following: What happens if we consider an $n \times n$ matrix partitioned into $k \times k$ blocks, instead of 2×2

blocks? We had already considered this problem for special cases in [2]. Our results can be also found in [5, 6, 7]. Let F be an arbitrary field. Let $n, k, p_1, ..., p_k$ be positive integers such that $n = p_1 + \cdots + p_k$. Let

$$C = \begin{bmatrix} C_{1,1} & \cdots & C_{1,k} \\ \vdots & & \vdots \\ C_{k,1} & \cdots & C_{k,k} \end{bmatrix} \in F^{n \times n}. \tag{4}$$

Firstly, assume that all the blocks $C_{i,j}$ are of the same size. In [6], we showed that given an $n \times n$ matrix of the form (4) partitioned into $k \times k$ blocks of the same size $p \times p$, with entries in an arbitrary field F, it is always possible to prescribe $2k - 3$ blocks of the matrix and the eigenvalues in F, except if, either all the principal blocks are prescribed, or all the blocks of one row or column are prescribed. In these exceptional cases, we identify necessary and sufficient conditions for which it is possible to prescribe $2k - 3$ blocks of the matrix and the eigenvalues in F. Our results are the following.

Let F be an arbitrary field. Let k, p be positive integers and $n = kp$. Let $(r_1, s_1), ..., (r_{2k-3}, s_{2k-3})$ be distinct elements of $\{1, ..., k\} \times \{1, ..., k\}$. Let $A_{r_i s_i} \in F^{p \times p}$, $i \in \{1, ..., 2k-3\}$. Let $c_1, ..., c_n \in F$.

We start by studying the exceptional cases. In the following result we describe the eigenvalues of a matrix of the form (4), when all the blocks of one row or column are prescribed. Since simple similarity transformations, like simultaneous permutations of rows and columns or transposition, do not alter the eigenvalues of the matrix, we can assume, without loss of generality, that all the blocks of the first row are prescribed.

Theorem 18 [6]**.** *Suppose that all the blocks of the first row are prescribed. Let* $f_1(x) | \cdots | f_p(x)$ *be the invariant factors of*

$$[xI_p - A_{1,1} \quad -A_{1,2} \cdots \quad -A_{1,k}].$$

There exists a matrix of the form (4), *where the blocks* $C_{i,j}$ *are of size*

$p \times p$, with eigenvalues $c_1, \ldots, c_n$, such that $C_{r_i, s_i} = A_{r_i, s_i}$, $i \in \{1, \ldots, 2k - 3\}$, if and only if

$$f_1(x) \cdots f_p(x) \mid (x - c_1) \cdots (x - c_n).$$

In the following theorem we study the case where all the principal blocks are prescribed.

Theorem 19 [6]**.** *Suppose that all the principal blocks are prescribed. Then there exists a matrix of the form* (4), *where the blocks $C_{i,j}$ are of size $p \times p$ with eigenvalues $c_1, \ldots, c_n$, such that $C_{r_i, s_i} = A_{r_i, s_i}$, $i \in \{1, \ldots, 2k - 3\}$, if and only if*

$$\sum_{i=1}^{k} \operatorname{tr} A_{i,i} = \sum_{j=1}^{n} c_j.$$

Finally, we establish a result that shows that it is always possible to prescribe $2k - 3$ blocks of the matrix, simultaneously with the eigenvalues, except in the previous cases.

Theorem 20 [6]**.** *Suppose that at least one principal block is free and at least one block in each row and each column is free. Then there exists a matrix of the form* (4), *where the blocks $C_{i,j}$ are of size $p \times p$, with eigenvalues $c_1, \ldots, c_n$, such that $C_{r_i, s_i} = A_{r_i, s_i}$, $i \in \{1, \ldots, 2k - 3\}$.*

Note that for $p = 1$, our answer generalizes the result established by Hershkowitz in [11].

Notice that the number $2k - 3$ cannot be increased without any additional condition. For example, given a matrix of the form (4), from the interlacing inequalities for the invariant factors [24, 29], it follows that some of the roots of the invariant factors of

$$\begin{bmatrix} -C_{1,2} & -C_{1,3} & \cdots & -C_{1,k} \\ xI_p - C_{2,2} & -C_{2,3} & \cdots & -C_{2,k} \end{bmatrix}$$

are eigenvalues of (4).

Still considering all the blocks $C_{i,j}$ of the same size, in [5] we described the possible characteristic polynomials of a matrix of the form (4) partitioned into $k \times k$ blocks of the same size $p \times p$, when some of the blocks are prescribed and the others vary. Our answer shows that it is always possible to prescribe $k-1$ blocks of the matrix and the characteristic polynomial, except in an exceptional case. In this exceptional case, we identify a necessary and sufficient condition under which the problem has solution.

Let k, p be positive integers and $n = kp$. Let $(r_1, s_1), \dots, (r_{k-1}, s_{k-1})$ be distinct elements of $\{1, \dots, k\} \times \{1, \dots, k\}$. Let $A_{r_i, s_i} \in F^{p \times p}$, $i \in \{1, \dots, k-1\}$. Let $f(x) \in F[x]$ be a monic polynomial of degree n.

Theorem 21 [5]. *Consider the following exceptional case*:

(E) *All the* $k-1$ *nonprincipal blocks of one row or column of* (4) *are prescribed equal to* 0.

If (E) *holds, then there exists a matrix of the form* (4), *with characteristic polynomial* $f(x)$ *and* $C_{r_i, s_i} = A_{r_i, s_i}$, $i \in \{1, \dots, k-1\}$, *if and only if* $f(x)$ *has a divisor of degree* p.

If (E) *is not satisfied, then there exists a nonderogatory matrix of the form* (4), *with characteristic polynomial* $f(x)$ *and* $C_{r_i, s_i} = A_{r_i, s_i}$, $i \in \{1, \dots, k-1\}$.

Note that for $p = 1$, we obtain the result established by Dias da Silva in [8].

When we consider the more general problem, where the blocks are not necessarily with the same size, the situation becomes more difficult. In fact if the prescribed positions correspond to "large" submatrices, then there are necessary interlacing inequalities involving invariant factors [24, 29]. We are not able to prove the sufficiency of these conditions. In the next result we study a particular situation: we describe the possible eigenvalues of a matrix of the form (4), where the blocks $C_{1,1}, \dots, C_{k,k}$

are square, not necessarily with the same size, when a diagonal of blocks is prescribed.

Theorem 22 [7]**.** *Let* $n, k, p_1, \ldots, p_k$ *be positive integers such that* $k \geq 3$ *and* $n = p_1 + \cdots + p_k$, *let* σ *be a permutation of* $\{1, \ldots, k\}$, *let* $A_{i,\sigma(i)} \in F^{p_i \times p_{\sigma(i)}}$, *for every* $i \in \{1, \ldots, k\}$, *and let* $c_1, \ldots c_n \in F$. *Then there exists a matrix of the form* (4) *with eigenvalues* $c_1, \ldots, c_n$, *such that* $C_{i,\sigma(i)} = A_{i,\sigma(i)}$, *for every* $i \in \{1, \ldots, k\}$, *if and only if the following conditions are satisfied*:

(a) *If* σ *is the identity*, *then* $\operatorname{tr}(A_{1,1} + \cdots + A_{k,k}) = c_1 + \cdots + c_n$;

(b) *If there exists* $i \in \{1, \ldots, k\}$ *such that* $\sigma(i) = i$, $p_i > n/2$ *and* $f_1(x) | \cdots | f_{p_i}(x)$ *are the invariant polynomials of* $A_{i,i}$, *then*

$$f_1(x) \cdots f_{2p_i - n}(x) | (x - c_1) \cdots (x - c_n).$$

Note that we present the solution for $k \geq 3$, since the case $k = 2$ had already been studied by Silva. In [26] the author gave the complete answer if the main diagonal is prescribed (see Theorem 10), and in [25] Silva also presented the complete solution, when the nonprincipal diagonal is prescribed (see Corollary 14).

Our main goal is to describe the possible eigenvalues or the characteristic polynomial of (4), when we consider k prescribed blocks arbitrarily positioned. As we had already observed, if the prescribed positions correspond to “large” submatrices, there are necessary interlacing inequalities involving the invariant factors [24, 29]. The proof of the sufficiency of these conditions can be very hard. This general problem is still open. We start by studying case $k = 3$, to give us an idea for the more general question.

Let F be an arbitrary field. Let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$ and let

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \in F^{n \times n}, \tag{5}$$

be a partitioned matrix where $C_{i,j} \in F^{p_i \times p_j}$, $i, j \in \{1, 2, 3\}$. Our aim is to describe the possible eigenvalues of (5), when some of its blocks are prescribed and the others are free. Clearly, we obtain many cases, according to the different positions of the prescribed blocks. We split our answer in different results, according to the location of the prescribed blocks.

Theorem 23 [3]. *Let* $c_1, \ldots, c_n \in F$. *Let* $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$ *and* $C_{2,1} \in F^{p_2 \times p_1}$. *Then there exist* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ *such that the matrix of the form* (5) *has eigenvalues* $c_1, \ldots, c_n$.

Theorem 24 [4]. *Let* $c_1, \ldots, c_n \in F$. *Let* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$. *Let* $\alpha_1 | \cdots | \alpha_{p_1}$ *be the invariant factors of*

$$[xI_{p_1} - C_{1,1} \quad -C_{1,2}] \tag{6}$$

and let $\beta_1 | \cdots | \beta_{p_3}$ *be the invariant polynomials of* $C_{3,3}$. *Consider the following conditions*:

(a) *If* $p_1 > p_3$, *then* $\alpha_1 \cdots \alpha_{p_1 - p_3} \mid (x - c_1) \cdots (x - c_n)$;

(b) *If* $p_3 > p_1 + p_2$, *then* $\beta_1 \cdots \beta_{p_3 - p_1 - p_2} \mid (x - c_1) \cdots (x - c_n)$.

Then there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,1} \in F^{p_2 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$ *such that the matrix of the form* (5) *has eigenvalues* $c_1, \ldots, c_n$ *if and only if either* (a) *or* (b) *is satisfied.*

Theorem 25 [1]. *Let* $c_1, \ldots, c_n \in F$. *Let* $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$. *Let* $\alpha_1 | \cdots | \alpha_{p_1}$ *be the invariant factors of* (6). *Then there exist* $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,1} \in F^{p_2 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{3,1} \in$

$F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ *such that the matrix of the form* (5) *has eigenvalues* $c_1, \ldots, c_n$ *if and only if*

$$\alpha_1 \cdots \alpha_{p_1 - p_3} \mid (x - c_1) \cdots (x - c_n).$$

(*Make convention that* $\alpha_1 \cdots \alpha_{p_1 - p_3} = 1$ *if* $p_1 \leq p_3$).

Now suppose that the blocks $C_{1,1}$, $C_{1,2}$, $C_{3,1}$ are prescribed and the others are free. According to the type of conditions that appear in the studied cases $(k = 2,\ k \geq 3)$, it seems natural to think that the solution of this problem is the following.

Let $\alpha_1 | \cdots | \alpha_{p_1}$ be the invariant factors of (6) and let $\beta_1 | \cdots | \beta_{p_1}$ be the invariant factors of

$$\begin{bmatrix} xI_{p_1} - C_{1,1} \\ -C_{3,1} \end{bmatrix}. \tag{7}$$

Then there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,1} \in F^{p_2 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (5) has eigenvalues $c_1, \ldots, c_n$ if and only if

$$\alpha_1 \cdots \alpha_{p_1 - p_3} \beta_1 \cdots \beta_{p_1 - p_2} \mid (x - c_1) \cdots (x - c_n). \tag{8}$$

(With the convention that $\alpha_1 \cdots \alpha_{p_1 - p_3} = 1$ if $p_1 \leq p_3$ and $\beta_1 \cdots \beta_{p_1 - p_2} = 1$ if $p_1 \leq p_2$).

Curiously, condition (8) is not necessary. It is not hard to find counterexamples. For example, suppose that $p_1 = 6$, $p_2 = p_3 = 1$ and $C_{1,1} = I_6$, $C_{1,2} = 0 \in F^{6 \times 1}$, $C_{3,1} = 0 \in F^{1 \times 6}$ and suppose that there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,1} \in F^{p_2 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (5) has eigenvalues $c_1, \ldots, c_n$. Clearly, the invariant polynomials of $C_{1,1}$ are $\gamma_1 = \cdots = \gamma_6 = x - 1$. Consequently, the invariant factors of (6) are

$\alpha_1 = \cdots = \alpha_6 = x - 1.$ It is also clear that the invariant factors of (7) are $\beta_1 = \cdots = \beta_6 = x - 1.$ In this case, condition (8) means that:

$$\alpha_1 \cdots \alpha_5 \beta_1 \cdots \beta_5 = (x-1)^{10} \mid (x - c_1) \cdots (x - c_7),$$

which is impossible. Therefore, condition (8) is not necessary. It remains open if this condition is sufficient.

4. Concluding Remarks

The general problem of describing the possible eigenvalues or the characteristic polynomial of a matrix of the form (4) when $k > 3,$ and some of its blocks are prescribed and the remaining are unknown, is still open. We start by establishing some results when all the blocks are of the same size. Concerning the more general case, where the blocks are not necessarily of the same size, we present a solution in a particular situation: when a diagonal of blocks is prescribed.

Our main goal of describing the possible eigenvalues or the characteristic polynomial of (4), when k prescribed blocks are arbitrarily positioned, is still open. In order to give some ideas for this general question, we provide some answers for the case $k = 3.$ Notice that when the prescribed positions correspond to "large" submatrices, there are necessary interlacing inequalities involving invariant factors [24, 29]. The technique used to prove these inequalities can be very hard. This type of condition is the main hindrance in these problems. However, as the studied cases show, these conditions cannot be avoided .

In particular, the description of the eigenvalues or the characteristic polynomial of a matrix of the form (5), when $C_{1,1}, C_{1,2}, C_{3,1}$ are prescribed, is still open. In this paper we show that condition (8) is not necessary. We believe that the solution for this problem will contemplate another condition involving the invariant factors of (6) and the invariant factors of (7). At this moment, this problem is not yet solved. Note that the solution for this case, will include the solution of Problem (P_7) mentioned in Section 3, which is the most difficult problem for the case $k = 2.$ Notice that after more than 30 years, this problem remains without solution. For this case, only trivial results are known.

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